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I.—ON THE VIBRATIONS OF A MUSICAL STRING.*

THE tension of the string being supposed very great compared with its weight, in the position of equilibrium, the string will be as nearly as possible rectilinear; we shall here suppose it accurately so.

Let ξ be the distance of any point of the string from one extremity when the string is at rest, and let x, y, z be the three coordinates of the same point at the time t, x having the same origin and direction with ξ, and y and z being perpendicular to x and to each other.

It is plain that x, y, z are functions of ξ and t; hence a point, which when the string is at rest is situated at a distance $\xi + \delta\xi$ from the origin, will, at the time t, have its three coordinates respectively equal to

$$x + \frac{dx}{d\xi}\delta\xi, \quad y + \frac{dy}{d\xi}\delta\xi, \quad z + \frac{dz}{d\xi}\delta\xi;$$

$$\frac{dx}{d\xi}, \quad \frac{dy}{d\xi}, \text{ and } \frac{dz}{d\xi}$$

representing the partial differential coefficients with respect to ξ, t being regarded as constant.

Hence, if δs be the distance of the two points of the string above-mentioned at the time t, and α, β, γ the angles which δs makes with the three axes, we shall have

* From a Correspondent.

$$\delta s = \delta\xi \sqrt{\frac{dx^2}{d\xi^2} + \frac{dy^2}{d\xi^2} + \frac{dz^2}{d\xi^2}},$$

$$\cos \alpha = \frac{\frac{dx}{d\xi}\,\delta\xi}{\delta s} = \frac{\frac{dx}{d\xi}}{\sqrt{\frac{dx^2}{dy^2} + \frac{dy^2}{d\xi^2} + \frac{dz^2}{d\xi^2}}},$$

$$\cos \beta = \frac{\frac{dy}{d\xi}\,\delta\xi}{\delta s} = \frac{\frac{dy}{d\xi}}{\sqrt{\frac{dx^2}{d\xi^2} + \frac{dy^2}{d\xi^2} + \frac{dz^2}{d\xi^2}}},$$

$$\cos \gamma = \frac{\frac{dz}{d\xi}\,\delta\xi}{\delta s} = \frac{\frac{dz}{d\xi}}{\sqrt{\frac{dx^2}{d\xi^2} + \frac{dy^2}{d\xi^2} + \frac{dz^2}{d\xi^2}}}.$$

The above expressions are general whatever be the extent of the vibrations. But if we limit our investigation to the case where the vibrations are extremely small, putting $x = \xi + u$, the quantities u, y, z, as well as their differential coefficients, are all extremely small. Hence, writing for $\frac{dx}{d\xi}$ its value $1 + \frac{du}{d\xi}$, and neglecting such magnitudes as may legitimately be neglected consistently with this hypothesis, we shall have very approximately,

$$\delta s = \delta\xi \left(1 + \frac{du}{d\xi}\right),$$

$$\cos \alpha = 1, \quad \cos \beta = \frac{dy}{d\xi}, \quad \cos \gamma = \frac{dz}{d\xi}.$$

Hence, if T_1 be the tension of the string at the time t at the former of the two points of the string under consideration, or at the extremity of δs nearest the origin of the coordinates, the three components $T_1 \cos \alpha$, $T_1 \cos \beta$, $T_1 \cos \gamma$, will be very nearly expressed by

$$T_1, \quad T_1 \frac{dy}{d\xi}, \quad \text{and} \quad T_1 \frac{dz}{d\xi}.$$

At the other extremity of δs these three components will be respectively increased by

$$\frac{dT_1}{d\xi}\,.\,\delta\xi, \quad \frac{d}{d\xi}\left(T_1 \frac{dy}{d\xi}\right).\,\delta\xi, \quad \text{and} \quad \frac{d}{d\xi}\left(T_1 \frac{dz}{d\xi}\right).\,\delta\xi;$$

which three expressions are therefore the moving forces which tend to move the particle δs parallel to the three axes, and to *increase* the coordinates x, y, z.

Let W be the weight of a *unit of length* of the string in the state of rest. Then $\frac{W}{g}$ is the mass of that portion of string: consequently $\frac{Wd\xi}{g}$ is the mass of the portion $\delta\xi$, which is the same as that of δs at the time t. Hence, instituting the equation of motion, and writing for $\frac{d^2x}{dt^2}$ its equivalent $\frac{d^2u}{dt^2}$, we get

$$\frac{d^2u}{dt^2} = \frac{g}{W} \cdot \frac{dT_1}{d\xi},$$

$$\frac{d^2y}{dt^2} = \frac{g}{W} \cdot \frac{d}{d\xi}\left(T_1 \frac{dy}{d\xi}\right),$$

$$\frac{d^2z}{dt^2} = \frac{g}{W} \cdot \frac{d}{d\xi}\left(T_1 \frac{dz}{d\xi}\right).$$

Let T be the tension of the whole string when at rest; then T_1 is manifestly a function of the ratio $\frac{\delta s}{\delta\xi}$, which reduces itself to T when $\delta s = \delta\xi$, or when $\frac{\delta s}{\delta\xi} - 1 = 0$; that is, when $\frac{du}{d\xi} = 0$, $\left\{\text{since } \delta s = \delta\xi \cdot \left(1 + \frac{du}{d\xi}\right)\right\}$. If, therefore, we regard T_1 expanded in powers of $\frac{\delta s}{d\xi} - 1$ or $\frac{du}{d\xi}$, we shall have $T_1 = T + Q\,\frac{du}{d\xi}$, neglecting the higher powers of $\frac{du}{d\xi}$. The constant Q must be determined experimentally.

Now it appears by experiment, that within certain limits the increase of tension is proportional to the increase of length. Suppose the unit of length, whose tension is originally T, to be pulled out to the length $1 + \lambda$, and let P, its increase of tension, be observed, the portion $\delta\xi$ would be extended to the length $\delta\xi \cdot \frac{1+\lambda}{1}$; but when $\delta\xi$ is extended to the length δs, the increase of tension is $T_1 - T$; therefore

$$\frac{T_1 - T}{P} = \frac{\delta s - \delta\xi}{\delta\xi\,(1 + \lambda) - \delta\xi} = \frac{1}{\lambda} \cdot \left(\frac{\delta s}{\delta\xi} - 1\right);$$

$$\therefore T_1 = T + \frac{P}{\lambda}\left(\frac{\delta s}{\delta\xi} - 1\right)$$

$$= T + \frac{P}{\lambda} \cdot \frac{du}{d\xi};$$

hence $Q = \frac{P}{\lambda}$, and is therefore known.

If in the three equations of motion we write for T_1 its value $T + Q \cdot \frac{du}{d\xi}$, and neglect the products $\frac{dy}{d\xi} \cdot \frac{du}{d\xi}$ and $\frac{dz}{d\xi} \cdot \frac{du}{d\xi}$, they will become

$$\frac{d^2u}{dt^2} = b^2 \cdot \frac{d^2u}{d\xi^2},$$

$$\frac{d^2y}{dt^2} = a^2 \cdot \frac{d^2y}{d\xi^2},$$

$$\frac{d^2z}{dt^2} = a^2 \cdot \frac{d^2z}{d\xi^2}.$$

where $b^2 = \frac{Q}{W} g$, and $a^2 = \frac{T}{W} g$.

The form of these three equations of motion shows, that the vibrations parallel to the three axes take place independently of each other.

Let us first consider those parallel to the axes of y, which are given by the equation

$$\frac{d^2y}{dt^2} = a^2 \frac{d^2y}{dx^2},$$

in which ξ has been changed into x: no confusion will arise from this change, as we shall not again use the latter symbol in its original signification. The integral of the last equation is

$$y = f(x + at) + F(x - at),$$

the curves denoted by the functions f and F may be any whatever, continuous or discontinuous: the only limitation to which they are subject is, that the curvature shall in no point be suddenly interrupted, or, in other words, that the tangent at any point of the curve shall nowhere make a finite angle with the tangent at a point immediately adjacent.

The velocity parallel to y at any point, is determined by the equation

$$\frac{dy}{dt} = af'(x + at) - aF'(x - at).$$

Let us suppose that the string is initially displaced from its position of equilibrium, for a moment kept at rest in that displaced position, and then abandoned to itself; and let t be measured from the moment of this abandonment.

Then since, when $t = 0$, every point is at rest, the last equation gives

$$f'(x) = F'(x), \text{ whence } F(x) = 2A + f(x),$$

2A being an arbitrary constant; consequently

$$y = A + f(x + at) + A + f(x - at),$$

or writing $\frac{1}{2}\phi(v)$ for the form $A + f(v)$,

$$y = \tfrac{1}{2}\phi(x + at) + \tfrac{1}{2}\phi(x - at).$$

The form of ϕ is known when the initial shape of the curve is given; for making $t = 0$ in the last equation, we get

$$y = \phi(x).$$

Let l be the distance between the two fixed extremities of the string: the last equation gives the values of $\phi(v)$ from $v = 0$ to $v = l$. The condition, that the two extremities are fixed, will enable us to determine the value of $\phi(v)$ from $v = -\infty$ to $v = +\infty$. For since the first extremity is fixed for which $x = 0$, we must have

$$0 = \tfrac{1}{2}\phi(at) + \tfrac{1}{2}\phi(-at)$$

for all positive values of t, and consequently

$$\phi(-v) = -\phi(v)$$

for all values of v. The condition that the other extremity is fixed gives us in like manner

$$0 = \tfrac{1}{2}\phi(l + at) + \tfrac{1}{2}\phi(l - at)$$

for all positive values of t, and consequently

$$\phi(l + v) = -\phi(l - v)$$

for all positive values of v. Writing in the last equation $l + v$ in the place of v, we get

$$\phi(2l + v) = -\phi(-v),$$

and therefore $\phi(2l + v) = \phi(v)$.

Hence, the value of $\phi(v)$ from $v = 0$ to $v = l$ being given by the initial shape of the curve, its value from $v = l$ to $v = 2l$ will be given by the equation

$$\phi(l + v) = -\phi(l - v):$$

again, from $v = 2l$ to $v = 4l$, $\phi(v)$ will be given by the equation

$$\phi(2l + v) = \phi(v);$$

and in the same manner, by successive substitutions in the last equation, the value of $\phi(v)$ may be determined for any positive value of v. But $\phi(v)$ being known for all positive values of v, will also be known for all negative values, by means of the equation

$$\phi(-v) = -\phi(v).$$

Hence $\phi(v)$ is known from $v = -\infty$ to $v = +\infty$.

Having thus determined the values of the function $\frac{1}{2}\phi(v)$ for all values of v, positive or negative, the values of $\frac{1}{2}\phi(x + at)$ and $\frac{1}{2}\phi(x - at)$ will be known for all values of x and t; and thus the equation

$$y = \tfrac{1}{2}\phi(x + at) + \tfrac{1}{2}\phi(x - at)$$

determines the position of every point of the string at any assigned time.

It may be observed, that if we construct the curve $y = \frac{1}{2}\phi(x)$ from $x = -\infty$ to $x = \infty$, the equation

$$\phi(l + v) = -\phi(l - v)$$

shows, that from $x = l$ to $x = 2l$ the curve is the same as from $x = 0$ to $x = l$, except that it is inverted, both with respect to right and left, and with respect to up and down. Again, from $x = 2l$ to $x = 3l$ it is the same as from $x = 0$ to $x = l$, as appears from the equation

$$\phi(2l + v) = \phi(v).$$

From $x = 3l$ to $x = 4l$, since

$$\phi(3l + v) = \phi(l + v) = -\phi(l - v).$$

the curve will be inverted, as between $x = l$ and $x = 2l$; and the same succession of direct and inverted curve will recur *ad inf.*

On the negative side of the origin, the equation

$$\phi(-v) = -\phi(v)$$

shows, that the same alternation of curves in the inverted and direct position will succeed each other *ad inf.* as on the positive side.

The quantities $x + at$ and $x - at$, represent for a given value of x two lengths of line, the one increasing and the other decreasing with the uniform velocity a. And since $\frac{1}{2}\phi(x + at)$ remains of the same value when x becomes $x - h$ and t becomes $t + \frac{h}{a}$, we see that $\frac{1}{2}\phi(x + at)$ represents an ordinate, which is transferred from the abscissa x to the abscissa $x - h$ in a time $\frac{h}{a}$, and consequently with a velocity $h \div \frac{h}{a}$ or (a). In like manner, since $\frac{1}{2}\phi(x - at)$ remains of the same value when x becomes $x + h$ and t becomes $t + \frac{h}{a}$, $\frac{1}{2}\phi(x - at)$ represents an ordinate, which is transferred from the abscissa x to the abscissa $x + h$ in the time $\frac{h}{a}$, and therefore with the velocity (a).

Since what has just been said is true of all values of x for the same values of the time, namely t and $t + \frac{h}{a}$, it appears that $\frac{1}{2}\phi(x + at)$ represents a form of curve or undulation which is transferred towards the *negative* infinity with the uniform velocity (a); and $\frac{1}{2}\phi(x - at)$ represents a curve or undulation of exactly the same shape as the former, transferred towards the *positive* infinity with the same uniform velocity. The curve which results from the superposition of these two undulations at any instant, is the momentary shape of the vibrating string.

When $t = \frac{2l}{a}$, $y = \frac{1}{2}\phi(x + 2l) + \frac{1}{2}\phi(x - 2l)$,

$$\begin{aligned} \text{but } \phi(x + 2l) &= \phi(x), \\ \text{and } \phi(x - 2l) &= -\phi(2l - x) = -\phi\{l + (l - x)\} \\ &= \phi\{l - (l - x)\}, \ \{\text{since } \phi(l + v) = -\phi(l - v)\} \\ &= \phi(x); \\ \therefore y &= \phi(x). \end{aligned}$$

Hence the string returns to its original shape after a time $\frac{2l}{a}$, which is therefore the time of a complete vibration.

The best method of exhibiting to the eye the changes of shape which the string passes through in the course of each vibration, is actually to draw the curves one below the other, which must be superposed in order to produce the instantaneous form of the string.

Figure (1) represents one vibration divided into four distinct phases of equal duration. The upper of the three lines in each phase represents the string itself vibrating between its extreme points A, B, which are fixed. The second and third lines represent the two constituent forms $\frac{1}{2}\phi(x + at)$ and $\frac{1}{2}\phi(x - at)$.

In the first phase, where $t = 0$, each of the constituent forms has for its equation $y = \frac{1}{2}\phi(x)$, and will be drawn by taking the ordinates half the size of those of the initial figure represented in the upper line, whose equation is $y = \phi(x)$.

In the second line, from $x = l$ to $x = 2l$, the same constituent form is repeated in an inverted position; and from $x = 2l$ to $x = 3l$, the same form is again repeated in the original position.

In the third line the same constituent form is drawn, in its direct position under the vibrating string, then in its inverted position from $x = 0$ to $x = -l$, afterwards in its erect position from $x = -l$ to $x = -2l$.

This construction being made, the position of any point P of the line is easily determined at any time t as follows.

Draw the corresponding ordinates PM, P_1M_1, P_2M_2, in the original curve APB, and its two constituents $A_1P_1B_1$, $B_2P_2A_2$. Take M_1N_1 and M_2N_2, the first on the positive side of M_1 and the second on the negative side of M_2, making them each equal to (at), and draw the ordinates N_1Q_1, N_2Q_2. Then, taking

$$P'M = N_1Q_1 + N_2Q_2,$$

P′ will be the place of P at the time t.

It may be observed that N_1, according to the magnitude of t, may fall anywhere in M_1B_1, or M_1B_1 produced; and N_2 anywhere in M_2B_2, or M_2B_2 produced; and when the ordinates N_1Q_1, N_2Q_2 fall on the other side of the axis, they must be reckoned as negative in taking their algebraical sum, and P′M must be measured above or below the axis, as this algebraical sum is positive or negative.

But instead of setting off the ordinates N_1Q_1, N_2Q_2 along the *fixed* curves $A_1B_1C_1D_1$, $A_2B_2C_2D_2$, we may regard these curves as carried upon two separate sliders, and moved with the uniform velocity (a), the former towards the negative, and the latter towards the positive infinity. In this manner N_1Q_1 and N_2Q_2 would arrive under the point M of the string in the same instant; and to find the form of the whole string at that instant, we shall only have to take all the ordinates between A and B equal to the sum of the ordinates which fall immediately below them, paying proper attention to the signs as before directed.

We have thus an easy and simple method of exhibiting the string in any phase of vibration, and we shall presently see how it puts the subject of the harmonic sounds, and the reflexion of the undulations at the extremities of the string, under the clearest point of view.

In phase II. of fig. (1), each slider has been advanced through a space $\frac{1}{2}l$, and the figure of the string constructed by the rule laid down; and it may be observed in this, as in all the subsequent figures, how the condition of the fixity of the points A and B is satisfied by two equal and opposite ordinates of the constituent curves falling immediately below them.

In phase III. the sliders have advanced through a space (l), and we see that the shape of the string is the same as at first, only in an inverted position.

In phase IV. the sliders have advanced through a space $\frac{3}{2}l$, and the shape of the string is the same as in phase II.

In phase V. each slider has advanced through a space $2l$, and the string has reassumed its original form, so that this phase is merely a repetition of phase I.

The same succession of phases will recur in every subsequent vibration; and the time of each vibration is the time in which one of the sliders, moving with a velocity a, passes over a space $2l$; this time is therefore $\frac{2l}{a}$, as we have before found it to be in a less simple manner.

If the initial form of the string be such as is represented in fig. (2), where M, the middle point between the two fixed extremities A and B, divides the string into two equal segments, one being the exact *inverse* of the other; then constructing as before the constituent forms along A_1D_1 and A_2D_2, we see that the inverted form between B_1C_1 is exactly similar to the direct forms between A_1B_1 and between C_1D_1: and the same may be said of the forms along A_2D_2. Hence it is plain that the string will have reassumed its initial state when the sliders have advanced only through a space l, which takes place in the time $\frac{l}{a}$; that is, a time

half as great as in the former mode of vibration. The second phase here is the middle of the vibration, and the string assumes an inverted position, but it is not the *inverse* of its initial form in the same sense that phase III. is the inverse of phase I. in the former mode of vibration.

It is remarkable, that the middle point M, though fully at liberty to move, remains stationary during the whole time that the string vibrates; for if we take M_1 and M_2, the middle points of the constituent forms, and measure off M_1R_1 and M_2R_2 equal to each other, the former towards D_1 and the latter towards D_2, R_1 and R_2 will arrive under M at the same instant. Now, since the portion of curve $M_1Q_1B_1$ is by the hypothesis the inverse of the portion $B_2Q_2M_2$, the ordinate Q_1R_1 is equal and opposite to Q_2R_2. Hence M will still be in the same place in every position of the two sliders, so that each half of the string vibrates just as if M were a fixed point. This mode of vibration is called the first harmonic, and the sound it produces is the octave above that which is given by the string in its original mode of vibration.

In the same manner, if the string be initially divided at M and N, see fig. (3), into three equal segments, the segment between MN being the exact inverse of that between MN, and the segment between NB being the exact inverse of that between MN, and therefore a repetition of that between AM, it is easy to see that the string will reassume its initial form when the sliders have moved over the space $\frac{2}{3}l$, which will require a time equal to $\frac{2}{3}\frac{l}{a}$, or one-third the time of the original vibration. And it may be shown, as in the last case, that the points M and N remain stationary all the time the string vibrates, so that the portions AM, MN, NB vibrate as if each were a separate string. This mode of vibration is called the second harmonic, and the note it gives is the fifth above the note given by the first harmonic, or the twelfth above the fundamental note produced by the original vibration. In the figure three phases are exhibited, the third being a repetition of the first, and the sliders are advanced over a space $\frac{1}{3}l$ at each step.

There is no difficulty in extending these considerations to the harmonic sounds of the higher orders, and we shall dismiss this part of the subject with stating, that the third harmonic will give the double octave or fifteenth above the fundamental note, and the fourth will give the interval of a third above that, that is, the interval of a seventeenth above the fundamental note: and it is remarkable, that in the practice of music, these harmonic intervals constitute what is called the common chord, which the ear naturally admits as the most perfect combination of concordant sounds.

It is also worth observing, that these different modes of vibration may all exist at the same time in one and the same string, and may actually be distinguished as existing in the fundamental note by an accurate and musical ear. The mathematical explanation of this

depends on a well known property, which is common to all linear differential equations, or equations in which the differential coefficients of any order appear only under the first degree, being neither multiplied together nor raised to powers. The property consists in this, that if the equations $y=u$, $y=u_1$, $y=u_2$, &c. separately satisfy the proposed linear differential equation, the latter will be also satisfied by making

$$y = u + u_1 + u_2 + \ldots$$

In the present case this conclusion is an immediate consequence of the form of the general integral

$$y = f(x + at) + f(x - at).$$

For if

$$\begin{aligned} y &= \tfrac{1}{2}\phi(x + at) + \tfrac{1}{2}\phi(x - at), \\ y &= \tfrac{1}{2}\phi_1(x + at) + \tfrac{1}{2}\phi_1(x - at), \\ y &= \tfrac{1}{2}\phi_2(x + at) + \tfrac{1}{2}\phi_2(x - at), \\ \&c. &= \&c. \end{aligned}$$

be the integrals which separately correspond to the fundamental sound, the first harmonic, the second harmonic, &c.; and if we take

$$\tfrac{1}{2}f(v) = \tfrac{1}{2}\phi(v) + \tfrac{1}{2}\phi_1(v) + \tfrac{1}{2}\phi_2(v) + \ldots$$

$$\text{then } y = \tfrac{1}{2}f(x + at) + \tfrac{1}{2}f(x - at)$$

will be the integral which corresponds to a vibration in which all the preceding modes of vibration exist at once.

Let us now suppose the initial displacement to extend over a small portion only of the string.

Fig. (4) represents the vibration, in this case, in twelve different phases, the sliders advancing over a space $\frac{1}{6}l$ at each step.

Phase I. exhibits the string in its initial position.

In phase II. the original pulse has divided itself into two undulations, one moving towards A and the other towards B, which we will distinguish by the names undulation P and undulation Q.

In phase III. the undulation P is in the act of being reflected at the extremity A of the string, and Q is travelling on with the velocity (a) to the other extremity.

In phase IV. P is completely reflected, and Q has just reached the extremity of the string.

In phase V. P continues its course towards the extremity B, while Q is in the act of being reflected at that extremity.

In phase VI. P is travelling onwards towards B, and is meeting Q, which has just been reflected at B.

In phase VII. P and Q have reunited into one, which is the exact inverse of the original pulse in phase I.; the sliders have now travelled over a space l, and the semivibration is completed.

In phase VIII. the undulations have separated again; P has just reached the extremity B, and Q is travelling towards A.

In phase IX. P is in the act of being reflected at B, and Q continues its course towards A.

In phase X. P has been completely reflected at B, and Q has just reached the extremity A.

In phase XI. P is travelling towards A, and Q is in the act of being reflected at A.

In phase XII. P is just meeting Q, which is now completely reflected at A.

In phase XIII. P and Q have again coalesced, and the string is exactly as in phase I. The sliders have now moved over a space $2l$, and the whole vibration has been completed in a time $\frac{2l}{a}$.

Having thus fully discussed the vibrations parallel to y, we need not dwell upon those parallel to z, which will be precisely of the same nature and duration. It will be sufficient to remark, that the vibrations parallel to y may take place simultaneously with those parallel to z, and we may regard the initial form of the string as a curve of double curvature, whose projections on the planes of (y, x) and (z, x) determine the forms of the arbitrary functions. And it is easy to conclude that the string will reassume its original form after the time $\frac{2l}{a}$, this period being common both to the vibrations parallel to y and to those parallel to z.

The longitudinal vibrations parallel to ξ or x, which are given by the equation $\frac{d^2u}{dt^2} = b^2\left(\frac{d^2u}{dx^2}\right)$, lead to a discussion exactly similar to the preceding. They consist of a series of contractions and dilatations of the string, which travel along it with the uniform velocity b, and suffer reflexion at the ends of the string; and the time which elapses before the string returns to its initial state, or the time of a complete vibration, will be $\frac{2l}{b}$.

There will also be the same series of harmonic sounds, bearing the same relation to the fundamental sound as in the transverse vibrations.

Lastly, all the three species of vibrations, each with its concomitant harmonics, may coexist in the same vibrating string. But the longitudinal vibrations having quite a different period from the transverse, have no musical relation to them, though it is very possible that they may have a material effect in giving a quality to the sound, or in producing what the French term the "timbre" of the note. Experience shows that they produce sounds of a much higher pitch than the transverse vibrations, which requires that b be much greater than a, consequently Q much greater than T.

II.—ON A PROPERTY OF CERTAIN PARTIAL DIFFERENTIAL COEFFICIENTS.*

We propose to prove the following proposition—that if R be a function of three rectangular coordinates, x, y, z, such, that all the partial differential coefficients of R of the second order are constant and possible, then, a system of rectangular axes may always be found, such, that when the coordinates are referred to them, the functions corresponding to $\frac{d^2R}{dx\,dy}$ $\frac{d^2R}{dx\,dz}$ $\frac{d^2R}{dy\,dz}$ shall all vanish.

We premise the following: that in the case proposed, *any two* of the functions corresponding to $\frac{d^2R}{dx\,dy}$ $\frac{d^2R}{dx\,dz}$ $\frac{d^2R}{dy\,dz}$ may be made to vanish by transformation of coordinates.

If
$$\left.\begin{aligned} x_1 &= x\cos\theta - y\sin\theta \\ y_1 &= x\sin\theta + y\cos\theta \\ z_1 &= z \end{aligned}\right\}\ \ldots\ldots\ldots(a),$$
and
$$\begin{aligned} x' &= x_1, \\ y' &= y_1\cos\phi - z_1\sin\phi, \\ z' &= y_1\sin\phi + z_1\cos\phi, \end{aligned}$$
we shall find that
$$\frac{d^2R}{dx'\,dz'} = \frac{d^2R}{dx_1\,dy_1}\sin\phi + \frac{d^2R}{dx_1\,dz_1}\cos\phi,$$
$$\frac{d^2R}{dy'\,dz'} = \left(\frac{d^2R}{dy_1^{\,2}} - \frac{d^2R}{dz_1^{\,2}}\right)\sin\phi\cos\phi + \frac{d^2R}{dy'\,dz'}\cos 2\phi,$$
so that $\frac{d^2R}{dy'\,dz'}$, $\frac{d^2R}{dx'\,dz'}$ both vanish when
$$\tan 2\phi = -\frac{2\,\dfrac{d^2R}{dy_1\,dz_1}}{\dfrac{d^2R}{dy_1^{\,2}} - \dfrac{d^2R}{dz_1^{\,2}}},$$
$$\text{and}\quad \tan\phi = -\frac{\dfrac{d^2R}{dx_1\,dz_1}}{\dfrac{d^2R}{dx_1\,dy_1}};$$
in order to which we must have
$$\frac{\dfrac{d^2R}{dy_1\,dz_1}}{\dfrac{d^2R}{dy_1^{\,2}} - \dfrac{d^2R}{dz_1^{\,2}}} = \frac{\dfrac{d^2R}{dx_1\,dz_1}\cdot\dfrac{d^2R}{dx_1\,dy_1}}{\left(\dfrac{d^2R}{dx_1\,dy_1}\right)^2 - \left(\dfrac{d^2R}{dx_1\,dz_1}\right)^2},$$

* From a Correspondent.

$$\text{or } \left(\frac{d^2R}{dx_1\, dy_1} \cdot \frac{d^2R}{dy_1\, dz_1} - \frac{d^2R}{dy_1^2}\,\frac{d^2R}{dx_1\, dz_1} + \frac{d^2R}{dz_1^3}\,\frac{d^2R}{dx_1\, dz_1}\right)\frac{d^2R}{dx_1\, dy_1}$$
$$- \left(\frac{d^2R}{dx_1\, dz_1}\right)^2 \frac{d^2R}{dy_1\, dz_1} = 0.$$

From equations (a) we get

$$\frac{d^2R}{dy_1\, dz_1} = \frac{d^2R}{dx\, dz}\sin\theta + \frac{d^2R}{dy\, dz}\cos\theta,$$
$$\frac{d^2R}{dx_1\, dy_1} = \left(\frac{d^2R}{dx^2} - \frac{d^2R}{dy^2}\right)\sin\theta\cos\theta + \frac{d^2R}{dx\, dy}\cos 2\theta,$$
$$\frac{d^2R}{dx_1\, dz_1} = \frac{d^2R}{dx\, dz}\cos\theta - \frac{d^2R}{dy\, dz}\sin\theta,$$
$$\frac{d^2R}{dy_1^2} = \frac{d^2R}{dx^2}\sin^2\theta + \frac{d^2R}{dy^2}\cos^2\theta + \frac{d^2R}{dx\, dy}\sin 2\theta,$$
$$\frac{d^2R}{dz_1^2} = \frac{d^2R}{dz^2}.$$

By partially substituting these values, our condition may be made to assume the form

$$0 = \left.\begin{array}{l} \dfrac{d^2R}{dx^2}\,\dfrac{d^2R}{dy\, dz}\sin\theta - \dfrac{d^2R}{dy^2}\,\dfrac{d^2R}{dx\, dz}\cos\theta \\ - \left(\dfrac{d^2R}{dx\, dz}\sin\theta - \dfrac{d^2R}{dy\, dz}\cos\theta\right)\dfrac{d^2R}{dx\, dy} \\ + \left(\dfrac{d^2R}{dx\, dz}\cos\theta - \dfrac{d^2R}{dy\, dz}\sin\theta\right)\dfrac{d^2R}{dz^2} \end{array}\right\} \begin{array}{l} \dfrac{d^2R}{dx_1\, dy_1} \\ -\left(\dfrac{d^2R}{dx_1\, dz_1}\right)^2 \dfrac{d^2R}{dy_1\, dz_1}, \end{array}$$

which evidently is a cubic equation with respect to $\tan\theta$. Hence $\tan\theta$, and therefore θ, must have at least one real value, and therefore $\tan\phi$ and ϕ must have at least one real value; or the coordinates may be so taken as to cause the functions corresponding to $\frac{d^2R}{dx\, dz}$ $\frac{d^2R}{dy\, dz}$ to vanish. Let them be so taken, and let

$$x' = x\cos\omega - y\sin\omega,$$
$$y' = x\sin\omega + y\cos\omega,$$
$$z' = z;\ \text{then}$$

$$\frac{d^2R}{dx'\, dz'} = \frac{d^2R}{dx\, dz}\cos\omega - \frac{d^2R}{dy\, dz}\sin\omega = 0,$$
$$\frac{d^2R}{dy'\, dz'} = \frac{d^2R}{dx\, dz}\sin\omega + \frac{d^2R}{dy\, dz}\cos\omega = 0,$$
$$\frac{d^2R}{dx'\, dy'} = \left(\frac{d^2R}{dx^2} - \frac{d^2R}{dy^2}\right)\sin\omega\cos\omega + \frac{d^2R}{dx\, dy}\cos 2\omega;$$

which last will also vanish if

$$\tan 2\omega = -\frac{2\dfrac{d^2R}{dx\,dy}}{\dfrac{d^2R}{dx^2} - \dfrac{d^2R}{dy^2}},$$

a condition which may always be satisfied. Hence the proposition.

The reason why we arrive at a cubic equation for $\tan\theta$ is, obviously, because each one of *the final axes* satisfies the condition of the preliminary proposition.

We might prove, in a similar manner, the existence of the principal axes of bodies.

The above theorem embodies the analytical principle, upon which depends the similarity between the investigations relating to the axes of elasticity, and to diametral planes perpendicular to their ordinates in surfaces of the second order, as those investigations are conducted, respectively, in the first Number of this Journal, and in *Hymers's* Analyt. Geom., p. 141, 2nd edition. It will be readily seen how our theorem applies to the former of these. With respect to the latter, we would observe, that if

$$R = ax^2 + by^2 + cz^2 + a_1yz + b_1xz + c_1xy + a'x + b'y + c'z + d,$$

$$\frac{d^2R}{dy\,dz} = a_1, \quad \frac{d^2R}{dx\,dz} = b_1, \quad \frac{d^2R}{dx\,dy} = c_1;$$

and hence, by our proposition, the general equation of the second order may, by transformation, be put under the form

$$0 = ax^2 + by^2 + cz^2 + a'x + b'y + c'z + d.$$

Now, if $x = x - \dfrac{a'}{2a}$, this becomes

$$0 = ax^2 + by^2 + cz^2 + b'y + c'z - \frac{a'^2}{4a} + d;$$

hence the plane of yz is a diametral plane, perpendicular to its ordinates. If $a = 0$ the above fails; but as at least one of the quantities a, b, c must be finite, there must be at least one diametral plane perpendicular to its ordinates in every surface of the second order.

M.

III.—NEW METHOD OF SOLVING A BIQUADRATIC EQUATION.

It is known that the method of solving a recurring equation applies equally to equations of the form

$$x^4 + mpx^3 + m^2qx^2 + m^3px + m^4 = 0 \ldots\ldots (1),$$

for by dividing by x^2 and assuming $x + \frac{m^2}{x} = z$, it may be reduced to a quadratic in terms of z. If we represent this equation by

$$x^4 + Px^3 + Qx^2 + Rx + S = 0 \ldots\ldots(2);$$

then, since $P = mp$, $R = m^3p$, and $S = m^4$,

$$P^2S = R^2, \text{ or } S = \frac{R^2}{P^2} \ldots\ldots(3).$$

To determine the corresponding relation between the roots, let $\alpha, \frac{1}{\alpha}, \beta, \frac{1}{\beta}$, be the roots of the equation

$$x^4 + px^3 + qx^2 + px + 1 = 0,$$

then the roots of equation (1) will be

$$m\alpha, \frac{m}{\alpha}, m\beta, \frac{m}{\beta},$$

and since the products of the 1st and 2nd, and of the 3rd and 4th, are each equal to m^2, these four quantities are proportionals.

Now any biquadratic may be reduced to one whose roots are proportionals; for let a, b, c, d be the roots of any biquadratic, then $a-v, b-v, c-v, d-v$ will be proportionals of

$$(a-v)(d-v) = (b-v)(c-v)$$

$$\text{or } ad-(a+d)v = bc-(b+c)v,$$

$$\text{or } v = \frac{ad-bc}{a+d-(b+c)}.$$

This function admits of only three different values by the interchange of the four letters, and therefore v may be found by a cubic.

Let then

$$y^4 + qy^2 + ry + s = 0$$

be any biquadratic equation, deprived of its second term. Assume

$$y = x + v,$$

then the transformed equation is

$$x^4 + 4vx^3 + (6v^2+q)x^2 + (4v^3+2qv+r)x + v^4 + qv^2 + rv + s = 0 \ldots\ldots(4).$$

Determine v so that this equation shall be of the form

$$x^4 + Px^3 + Qx^2 + Rx + \frac{R^2}{P^2} = 0 \ldots\ldots(5);$$

$$\therefore v^4 + qv^2 + rv + s = \left(\frac{4v^3+2qv+r}{4v}\right)^2,$$

whence by reduction

$$v^3 + \frac{4s-q^2}{2r}v^2 - \tfrac{1}{2}qv - \tfrac{1}{8}r = 0.$$

Let a value of v be found from this cubic, and substituted in the coefficients of (4), it will then be of the form (5). It then only remains to divide (5) by x^2, and assume

$$x + \frac{R}{Px} = z,$$

when we shall have

$$z^2 + Pz + Q - \frac{2R}{P} = 0,$$

$$x^2 - 2x + \frac{R}{P} = 0,$$

whence four values of x may be found; and then, since

$$y = x + v,$$

the roots of the proposed equation are known.

We have supposed the second term taken away, only to render the coefficients of the reducing cubic simpler than they would otherwise have been; but the method would be exactly the same if the second term had not been taken away.

In this method, as in the others, it may be shown that the reducing cubic is not soluble by Cardan's method, except when the biquadratic has one pair of imaginary roots, and two real roots.

ff.

IV.—ON THE EXISTENCE OF A RELATION AMONG THE COEFFICIENTS OF THE EQUATION OF THE SQUARES OF THE DIFFERENCES OF THE ROOTS OF AN EQUATION.*

The equation of the squares of the differences of the roots gives the means of ascertaining whether any assigned equation has all its roots real; for if they be so, all the roots of the equation of differences must be real and positive, and consequently, by Descartes's rule of signs, all its coefficients must be alternately positive and negative. Accordingly, Waring applied it to this purpose, and in the *Philosophical Transactions* for 1763, gave the conditions of the reality of all the roots in equations of the fourth and fifth degree.

There will be as many conditions as there are coefficients—that is, as there are units in the degree of the equation of the squares of the differences; and therefore, for an equation of the n^{th} order the equation of the squares of the differences will of course be of

* From a Correspondent.

the $\frac{n \cdot n-1}{2}$th order. Thus, in the third order there would be three conditions, in the fourth six, and so on.

Lagrange remarked, however, that the number of conditions in these two cases reduced itself to two and three respectively; and he suggested, that a similar simplification might be possible in the ten conditions of the fifth order.

Sturm's theorem, which, however different in form, is still in substance intimately connected with the theory of the equation of the squares of the differences, enables us to ascertain the true number of independent conditions.

By this theorem, we deduce from a given equation $f(x)=0$ a series of n functions. These, with the original $f(x)$, make $n+1$ functions of x. We substitute in them the limits a and b, and the number of changes of sign lost between these limits is the number of real roots of n, which are to be found in this interval. Consequently we have only to write plus and minus infinity in Sturm's functions, to get the whole number of real roots belonging to the equation.

The signs of each function, when $\pm\frac{1}{0}$ is put for x, will of course be that of the first term, supposing each function to be arranged in a series of decreasing powers of x. And if the first term of each be positive, the series of signs at the superior limit will be all permanences, and at the inferior all alternations; that is, all the roots of the equation will be real.

Hence the reality of all the roots depends on the signs of $n+1$ terms. But of these, the sign of $f(x)$ is determined at the limits $\pm\frac{1}{0}$; so is that of $\frac{df(x)}{dx}$, which is Sturm's first function. Consequently there remain but $n-1$ terms on the sign of which the reality of roots depends. Instead, therefore, of $\frac{n \cdot n-1}{2}$ conditions, there are in reality but $n-1$.

Thus, in the equation of the third degree we find two conditions, in that of the fourth three, and so on, agreeing with what Lagrange found in these cases, and suspected in that of the fifth degree.

It is not very difficult to see why some of the coefficients of the equations of differences must be so connected with the rest, as not to give any independent condition.

In order to get an idea of this connection, let us imagine $n-1$ independent conditions, that is, $n-1$ functions of the coefficients of $f(x)=0$, which have a definite sign when all the roots are real. These functions are coefficients of the equation of the squares of the differences. Let them all become equal to zero, then we have

$n-1$ relations among the n roots, which will give every one of these roots, except one, in terms of that root. Now, the relations $b=a$, $c=a, \ldots k=a$, will fulfil our $n-1$ equations, because these evidently make all the coefficients of the equation of the squares of the differences vanish. Hence these are the relations implied in the $n-1$ equations we have assumed; and these would, it has just been said, make all the coefficients $=0$. Consequently, $n-1$ independent relations are all we can have, and it follows, that if $n-1$ independent coefficients of the equation of differences become $=0$, all will be so.

We may take the matter somewhat differently, still using the case of equal roots in $f(x)=0$ to show the relations among the coefficients of the equation of the squares of the differences.

If $f(x)=0$ has m equal roots, there will be $\frac{m(m-1)}{2}$ roots of the equation of the squares of the differences, or $\Delta=0$, equal to zero. Let $f(x)=0$ get another root equal to these m roots by any change in its coefficients, then there will be $\frac{(m+1)m}{2}$ roots in $\Delta=0$ equal to zero; the difference is m. Thus, a single fresh relation among the coefficients of $f(x)=0$ makes m coefficients of $\Delta=0$ vanish; for obviously the last coefficients of this equation disappear whenever it gets roots equal to zero.

We may easily see, too, that the constant function (the last in Sturm's process) is the same as the term independent of u in $\Delta=0$.

The equation $\Delta=0$ may, theoretically, be got by eliminating x between the two equations

$$f(x)=0 \text{ and } f'x+\frac{u}{2}f''x+\frac{u^2}{2\,.\,3}f'''x+\ldots u^{n-1}=0,$$

(*Lagrange*, p. 7); Δ will be the term independent of x: put, then, $u=0$, Δ reduces itself to its last term, and the process becomes simply that of finding the common measure of $f(x)$ and $f'(x)$, which, abstracting the changes of sign, is exactly Sturm's process; hence his term independent of x, will be "aux signes près" the constant term in $\Delta=0$.

The development of this idea would undoubtedly lead to the general theory of Sturm's method, and would make it more than a happy artifice, by showing its intimate connection with the equation of the squares of the differences. As is generally the case, the different ways in which the subject may be viewed, ultimately coalesce.

R. L. E.

V.—ON THE EXISTENCE OF BRANCHES OF CURVES IN SEVERAL PLANES.

By D. F. Gregory, B.A. Trin. Coll.

In tracing a curve expressed by an equation between two variables, it is customary to make use of negative as well as of positive values of the variables, but to reject those which are usually called impossible or imaginary. This practice was allowable so long as it was supposed that impossible quantities had no meaning in geometry; but if we once admit the possibility of interpreting them in this science, though not in arithmetic, we are bound in strict logic not to neglect them. Accordingly, the Abbé Buée, in his very ingenious paper in the *Philosophical Transactions* for 1806, in which he demonstrated the possibility of interpreting geometrically the symbol usually written $\sqrt{-1}$, showed that quantities affected by it corresponded to branches of the curve situate in a plane at right angles to the original plane. Professor Peacock is, I believe, the only author who agrees with Buée in this view of the nature of curves, although it seems difficult to show any reason why it should not be generally allowed. I believe that a name has had great influence in preventing its adoption—the word *imaginary* so frequently applied to the symbol $\sqrt{-1}$ appearing to make persons unwilling to believe that it could possibly admit of any interpretation. Yet, after all, the difference between it and the symbol $-$ is not so very great, both admitting of easy interpretation in the science of geometry, and neither, if considered independently, in the science of arithmetic, more especially if we consider them, as I have done elsewhere, as fractional powers of the symbol $+$, having peculiar properties depending on the fundamental definition of that symbol.

It appears to me, that if we once admit anything beyond what are called positive values of the variables, that is, pure arithmetical values wholly independent of the symbol $+$, there is no reason why we should confine ourselves to $-$ or $+^{\frac{1}{2}}$, since this is not differently circumstanced from any other power of $+$, analytically considered. I therefore hold, that we must either limit ourselves to the one quadrant formed by the positive axes, or we must be prepared to consider the curve as existing in several planes. Nor need it appear surprising, that by means of an equation between two variables, we are able to take into our view three dimensions, for the symbol $+$ is really equivalent to an angular coordinate, and therefore enables us to reach all points of space. In the following pages I propose, therefore, to extend farther than he has done, the principle introduced by Buée, and not confining myself to values of the variables of the form $+^{\frac{1}{2}}a$, to investigate the forms of curves, when we assume that the variables may be of the general form $+^{\frac{p}{q}}a$.

As a preliminary, we must consider what is the meaning of this expression when substituted in the equation to a curve $y = f(x)$. Since $+^{\frac{p}{q}}$ represents the turning of a line through an angle $\frac{2p\pi}{q}$, the expression $x = +^{\frac{p}{q}} a$ signifies that we are to measure a line whose length is a, along the axis of x, and then to turn the axis through an angle equal to $\frac{2p\pi}{q}$. But we may turn the axis in an infinite number of ways, which at first would make it appear that $x = +^{\frac{p}{q}} a$ would not give a definite point. But it is to be observed, that the axis of x is always to be perpendicular to that of y, so that it is only allowable to move the axis in a plane perpendicular to the plane of xy. If the substitution of $+^{\frac{p}{q}} a$ for x gives a value of y equal to $+^{\frac{r}{s}} b$, this implies that we have to measure a length b along y, and then to turn it through an angle equal to $\frac{2r\pi}{s}$, and the plane passing through the two new axes will be the plane of a branch of the curve formed by assigning all values to a from 0 to ∞, unless, which will not unfrequently happen, the value of a affects the index of $+$ in the expression for y, in which case every element of the curve will lie in a different plane from the contiguous one, and the curve will be one of double curvature. This, perhaps, will appear more clearly, if we illustrate it by an example in the case of the parabola, which is the simplest curve for our purpose. The equation to the parabola being

$$y^2 = mx,$$

if we put $+^{\frac{p}{q}} a$ for x, we find the value of y to be $+^{\frac{p}{2q}} m^{\frac{1}{2}} x^{\frac{1}{2}}$. Hence we see that the axis of y is to be turned round through an angle, which is one-half of that through which x is turned round; and for all the values which we may assign to a, if we leave p and q unchanged, the plane of the branch will remain unchanged, and the branch itself will be exactly similar to that in the plane xy, since the *numerical* relation between y and x is the same, whatever values we assign to p and q. By changing these last we obtain different branches in different planes; and as there is no limit to the values we may assign to them, it appears that the equation to the parabola, considered generally, represents a curve of an infinite number of branches, all passing through the origin, and situate in planes, such that the axis of x in any plane makes with the old axis twice the angle which the axis of y in that plane does with the old axis of y. As a particular case, we may take $\frac{p}{q} = \frac{1}{2}$, or make $x = -a$, whence

$$y = +^{\frac{1}{4}} (ma)^{\frac{1}{2}},$$

or the curve lies in a plane at right angles to the old plane.

The existence of this curve on the negative side of the axis of y, serves to explain an apparent anomaly which occurs in a very elementary problem. If we seek the equation to the locus of the intersection of a perpendicular from the focus of a parabola on the tangent, we obtain an equation which divides itself into two—the one representing the axis of y, which is the solution usually taken, the other furnishing only the focus. It seems strange that the focus should in any way be a solution of the problem, since the tangent of the positive branch of the curve never passes through that point. But if we consider the branch of the curve which lies in a plane perpendicular to the plane of xy, we see that all the tangents to that branch must pass through the positive axes of x, and consequently that one, or rather two, must pass through the focus, which thus is the point in which the tangent is met by a perpendicular from that point. Moreover, we find that the values of x and y, which belong to the focus, render the equation to the tangent of the form

$$y = (-)^{\frac{1}{2}} x + \beta,$$

showing that the tangent is in a plane at right angles to its original plane.

The form of the equation to the parabola renders it very easy to determine the value of y corresponding to that of x; but in the other curves of the second degree, though the investigation may not be so simple, we arrive at similar conclusions. Taking the equation to the ellipse referred to the centre,

$$y^2 = m^2 (a^2 - x^2),$$

$\left(\text{putting } \frac{b^2}{a^2} = m^2\right)$, we have for the value of y

$$y = m (a^2 - x^2)^{\frac{1}{2}}.$$

So long as $x < a$, whether we reckon x to be positive or negative, the value of y is possible, and the curve exists only in the plane of xy. If we make $x > a$, x being either positive or negative, we find y to be of the form $-^{\frac{1}{2}} mp$, showing that there is a branch of the curve in a plane at right angles to that of xy. Its form, it will be easily seen, is that of a hyperbola, since y increases with x, and becomes infinite when x is so, the relation between them being of the form

$$y = m (x^2 + a^2)^{\frac{1}{2}},$$

which is the equation to a hyperbola.

Hence, the vertices of the major axes of the ellipse are the vertices of two hyperbolas in a plane at right angles to the plane of the ellipse, and the ratio of the axes of which is the same as that of the axes of the ellipse. Also, since x and y are symmetrically involved in the equation to the ellipse, there must be a similar result

for the extremity of the minor axis, the only difference being, that the axes of the hyperbola will be reversed in position.

If, more generally, we suppose $x = +^{\frac{p}{q}} c$, the result is not so simple, for we have

$$y^2 = m^2 (a^2 - +^{\frac{2p}{q}} c^2);$$

from which we cannot directly determine the angle through which the axis of y is turned, corresponding to that through which x is supposed to be turned; but if we avail ourselves of the connexion which subsists between powers of $+$ and Demoivre's formula, we are able to determine the value of y.

Let $+^{\frac{p}{q}} = \cos\theta + -^{\frac{1}{2}} \sin\theta$, so that $\theta = \dfrac{2p\pi}{q}$, then

$$y^2 = m^2 \{a^2 - (\cos 2\theta + -^{\frac{1}{2}} \sin 2\theta)\, c^2\},$$
$$= m^2 \{a^2 - c^2 \cos 2\theta - -^{\frac{1}{2}} c^2 \sin 2\theta\};$$

whence $y = m\,(a^4 - 2a^2c^2 \cos 2\theta + c^4)^{\frac{1}{4}} (\cos\phi - -^{\frac{1}{2}} \sin\phi)$,

$$\text{where } \cos 2\phi = \frac{a^2 - c^2 \cos 2\theta}{(a^4 - 2a^2c^2\cos 2\theta + c^4)^{\frac{1}{2}}},$$

$$\sin 2\phi = \frac{c^2 \sin 2\theta}{(a^4 - 2a^2c^2 \cos 2\theta + c^4)^{\frac{1}{2}}}.$$

This result differs materially from that in the case of the parabola; for since the value of ϕ depends on c, the circulating function $\cos\phi - -^{\frac{1}{2}} \sin\phi$ depends on c, so that the angle through which the axis of y is to be turned, varies with the length of the abscissa, and the plane of the curve is constantly changing for every value of x, or in other words, the curve is one of double curvature.

When $c = 0$, $y = b$, and the curve passes through the extremities of the axis minor; and since c may be made infinite, the curve is infinite. Also, since x and y are symmetrically involved in the equation, there will be a similar curve passing through the extremities of the axis major. Hence, in addition to the hyperbolas already mentioned, the equation to the ellipse includes an infinite number of curves, with infinite branches passing through the extremities of the axes.

It is not necessary to consider the equation to the hyperbola, since it evidently leads to similar results. Nor is there much interest attached to the discussion of other more complicated curves, which I shall therefore omit with these exceptions—the curves of sines and cosines, and the Logarithmic curve. These I shall briefly touch on, as the form of their equations renders their discussion easy, and as there is considerable interest attached to them in a geometrical point of view.

If $y = a \sin x$,

then, making $x = +^{\frac{p}{q}}c$, we have

$$y = a \sin(+^{\frac{p}{q}}c);$$

and it remains to be considered what relation this bears to $\sin c$. If we suppose the sector of the circle whose angle is c to turn round the radius from which c is measured, it will be easily seen, that on turning it round through a circumference the angle will return to its original position, and so for any number of revolutions. Therefore, this operation of turning the sector through a circumference is subject to the laws of the symbol $+$, and may therefore be represented by it; and consequently, the operation of turning the sector through the $\frac{p}{q}$th part of a circumference will be properly represented by $+^{\frac{p}{q}}c$. But since the sine of the arc, being in the plane of the sector, is perpendicular to the axis of revolution, it will also be moved through the $\frac{p}{q}$th of a circumference, from which it follows, that

$$\sin(+^{\frac{p}{q}}c) = +^{\frac{p}{q}} \sin c.$$

But the cosine being measured along the axis of revolution, experiences no change corresponding to the change in the angle, so that we have

$$\cos(+^{\frac{p}{q}}c) = \cos c.$$

It may be observed, that these propositions are extensions of the two common theorems, that

$$\sin(-x) = -\sin x, \qquad \cos(-x) = \cos x.$$

Hence, in the case of the curve of sines, we have

$$y = +^{\frac{p}{q}} a \sin c,$$

which shows that the axis of y is to be turned through an angle equal to that through which the axis of x is turned. Hence, if we suppose the plane of xy to turn round an axis in its own plane, passing through the origin, and making equal angles with the two axes of x and y, in every position there will be a curve of sines exactly the same as that in the plane of xy. On the other hand, the equation to the curve of cosines being

$$y = a \cos x,$$

gives for $x = +^{\frac{p}{q}}c$,

$$y = a \cos(+^{\frac{p}{q}}c) = a \cos c,$$

which shows that the axis of y remains fixed; and if we suppose the plane of xy to turn round the axis of y, there will be a curve

of cosines, the same as that in xy, corresponding to every position of the plane.

The equation to the logarithmic curve is

$$y = \varepsilon^{nx}.$$

Let now $x = +^{\frac{p}{q}} a = a\,(\cos\theta + -^{\frac{1}{2}} \sin\theta)$ suppose.

$$\text{Then } y = \varepsilon^{na\,(\cos\theta + -^{\frac{1}{2}}\sin\theta)} = \varepsilon^{na\cos\theta} \,.\, \varepsilon^{na\sin\theta\, -^{\frac{1}{2}}}.$$

Now, since $+^{r} = \varepsilon^{2r\pi -^{\frac{1}{2}}}$, this gives us

$$y = (+)^{\frac{na\sin\theta}{2\pi}} \,.\, \varepsilon^{na\cos\theta};$$

thus determining the angle through which the axis of y is to be turned: and as this depends on a, the angle must (as in the case of the ellipse) vary with the length of the abscissa, so that the curve is not situate in one plane, but is a curve of double curvature. The absolute linear value of y, it will be easily seen, is less than that corresponding to the same linear value of x in the plane of xy, since the index of ε is reduced in the ratio of the cosine of θ to unity. It seems scarcely worth while further to discuss the nature of this curve, but having here adopted very different ideas concerning it from those promulgated by M. Vincent in Gergonne's *Annales des Mathematiques*, I think it necessary to state more at length my reasons for differing from that author.

In a paper in the preceding number of this Journal, I developed what I conceive to be the true theory of general logarithms, and I endeavoured to show that the impossible parts are really logarithms of the powers of $+$, the existence of which is generally overlooked. I pointed out that the formula of Mr. Graves, which agrees with that of M. Vincent, was derived from the supposition, that the base of the system of logarithms was of the form $+^{r} a$. This gives for the equation to the logarithmic curve,

$$y = (+^{r} a)^{x}.$$

Now M. Vincent assumes, that when x is fractional, y has as many values as there are units in the denominator of x; and when that is even, that two of these are possible—one positive and the other negative. Then he shows that the latter values do not form a continuous curve, but one of a kind which he calls *ponctuées*, whose nature is very peculiar, since, though the points are infinitely near to each other, yet we are able to draw an infinite number of straight lines between any two. This very strange result, so contrary to all our preconceived ideas of the nature of a curve, is sufficient, I think, to make us doubt the correctness of the method: but it is not very easy to point out the error, unless we employ the mode of considering the origin of the plurality of roots, which I have explained in the paper above referred to. According to that system, these various roots arise from our supposing a change to be

made in the value of r; and properly speaking there is no plurality of roots, but the nature of the quantity $+^r a$, whose root is taken, is indeterminate. Now here, in the case of the equation to the logarithmic curve, there is no indeterminateness, since r can only have one value. Any change in the value of r is a change in one of the constants of the equation to the curve, and consequently the equation no longer represents the same curve. Each of the points of the "*courbe ponctuée*" of M. Vincent, is really the point in which a certain curve meets the plane of xy. They are therefore wholly unconnected with each other, and cannot be reckoned as belonging to the same curve, either in a geometrical or analytical sense. This being granted, we perceive that there is no foundation for M. Vincent's very anomalous conclusion; and we are thus relieved from the necessity of believing in the existence of a species of line, of which we can hardly form a conception, and which has no sort of analogy to support it.

I said, that each point in the "*courbe ponctuée*" of M. Vincent belonged to a separate curve—it may be interesting to consider for a moment its nature. The equation

$$y = (+^r a)^x$$

gives $$y = +^{rx} a^x.$$

So long as x is an integer, this is the same as the simple equation

$$y = a^x,$$

and gives us the logarithmic curve in the plane of xy: but if we suppose x to be a fraction, it appears that the axis of y is to be turned through rx^{th} part of a circumference. As this angle varies with x, it appears that the curve is one of double curvature, and, as when $x=0$, $y=1$, it intersects the plane of xy at a distance 1 along the axis of y. But it may intersect it again; for if

$$x = \frac{1}{2r}, \quad +^{rx} = +^{\frac{1}{2}} = -,$$

and it cuts the plane of xy on the negative side of y; and if $x = \frac{1}{r}$, then $+^{rx} = +$, and it cuts the plane of xy on the positive side of y, meeting the curve traced in the plane of xy. On increasing x we shall obtain a similar curve which cuts the plane of xy, first above and then below the axis of x, the curve meeting the plane of xy whenever $rx = \frac{m}{2}$, m being either odd or even. It is needless to enter into the discussion of the complicated cases when x is of the form $+^{\frac{p}{q}} a$; more particularly as this paper has already exceeded its just limits. And I only will add, that these speculations derive their chief value from their bearing on the General Theory of the Science of Symbols. Practically, little attention will be paid to curves existing out of the plane of reference, since the

curves themselves do not come sufficiently under our eye to attract much interest. Perhaps the only way in which the existence of such curves is likely to be brought into notice, is in those cases where they serve to show the possibility of the interpretation of a solution of an equation. One case of this kind I have remarked on in this paper; several have been pointed out by Buée, and many I have no doubt will be added, when the attention of mathematicians is more particularly directed to the subject.

VI.—ON A PROPERTY OF THE HYPERBOLA.*

In a paper published in the *Philosophical Transactions* for 1836, the author, H. F. Talbot, Esq., has given a demonstration of a new theorem, connecting three arcs of the equilateral hyperbola. It may be stated thus: If in the equilateral hyperbola, referred to its asymptotes as axes, the sum of three abscissæ be zero, their product r, and the sum of the products of every pair $-\frac{r^2}{4}$, the sum of the arcs subtended by those abscissæ $= \frac{3}{4}r +$ const. The object of the present communication is, to prove that this theorem is included in a more general one, which gives a relation between three arcs of *any* hyperbola, referred to its asymptotes. The investigation furnishes a good example of Mr. Talbot's general method of finding the sum of a series of integrals, and may perhaps tempt some to consult the original memoir. There is another on the same subject in the *Transactions* for 1837, Part I.

The equation to the hyperbola, referred to its asymptotes, is

$$xy = \tfrac{1}{2}(a^2 + b^2);$$

and if a be the angle between the asymptotes,

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + 2\cos a\,\frac{dy}{dx}},$$

$$= \frac{1}{x^2}\sqrt{x^4 - 2m^2x^2\cos a + m^4},$$

where $m^2 = \frac{1}{2}(a^2 + b^2)$.

Hence the arc $= \int \frac{dx}{x^2}\sqrt{x^4 - 2m^2x^2\cos a + m^4}$.

Now, assume

$$\sqrt{x^4 - 2m^2x^2\cos a + m^4} = vx + m^2,$$

* From a Correspondent.

where v is a variable quantity. Squaring and dividing by x, we have

$$x^3 - (2m^2 \cos a + v^2)\, x - 2m^2 v = 0.$$

Let x, y, z be the roots of this equation, so that

$$x+y+z=0, \quad xy+xz+yz=-(2m^2 \cos a+v^2), \quad xyz=2m^2 v.$$

Then

$$\sqrt{x^4 - 2m^2x^2 \cos a + m^4} = vx + m^2,$$
$$\sqrt{y^4 - 2m^2y^2 \cos a + m^4} = vy + m^2,$$
$$\sqrt{z^4 - 2m^2z^2 \cos a + m^4} = vz + m^2.$$

Multiply these equations by $\frac{dx}{x^2}, \frac{dy}{y^2}, \frac{dz}{z^2}$, respectively, and add; then, denoting

$$\frac{1}{x^2}\sqrt{x^4 - 2m^2x^2 \cos a + m^4} \text{ by } \phi x,$$

$$\phi x \,.\, dx + \phi y \,.\, dy + \phi z \,.\, dz = v\left(\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}\right) + m^2\left(\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2}\right).$$

But

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = d \,.\log(xyz), \quad = d \,.\log 2m^2 v, \quad = \frac{dv}{v},$$

$$\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = -d \,.\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right),$$
$$= -d \,.\frac{xy + xz + yz}{xyz},$$
$$= d \,.\frac{2m^2 \cos a + v^2}{2m^2 v},$$
$$= \frac{dv}{2m^2} + \cos a\, d.\frac{1}{v};$$

therefore $\phi x \,.\, dx + \phi y \,.\, dy + \phi z \,.\, dz = \frac{3}{2} dv + m^2 \cos a \,.\, d \,.\frac{1}{v}$.

Integrating, we have

$$\int \phi x \,.\, dx + \int \phi y \,.\, dy + \int \phi z \,.\, dz = \frac{3}{2} v + \frac{m^2 \cos a}{v} + \text{const.}$$

Let the product of the abscissæ be r, which gives $v = \frac{r}{2m^2}$. Substituting for v this value, and for m^2 and $\cos a$ their values $\frac{1}{2}(a^2 + b^2)$ and $\frac{a^2 - b^2}{a^2 + b^2}$,

$$\int\phi x\,.\,dx+\int\phi y\,.\,dy+\int\phi z\,.\,dz=\frac{3}{2}\cdot\frac{r}{a^2+b^2}+\frac{1}{2}\cdot\frac{a^4-b^4}{r}+\text{const.}$$

The equations of condition become

$$x+y+z=0, \text{ and } xy+xz+yz=-(a^2-b^2)-\frac{r^2}{(a^2+b^2)^2}.$$

The theorem, therefore, may be stated thus: If in any hyperbola, referred to its asymptotes, the sum of three abscissæ be zero, their product r, and the sum of the products of every pair,

$$-(a^2-b^2)-\frac{r^2}{(a^2+b^2)^2},$$

then the sum of the corresponding arcs

$$=\frac{3}{2}\cdot\frac{r}{a^2+b^2}+\frac{1}{2}\cdot\frac{a^4-b^4}{r}+\text{const.}$$

If we make $a=b=1$, this reduces itself to Talbot's theorem.

It may be interesting to try a numerical example. For this purpose, integrating between the limits (x, x') of x, (y, y') of y, (z, z') of z, we have

$$\int_x^{x'}\phi x.dx+\int_y^{y'}\phi y.dy+\int_z^{z'}\phi z.dz=\frac{3}{2}\cdot\frac{r'-r}{a^2+b^2}+\frac{a^4-b^4}{2}\left(\frac{1}{r'}-\frac{1}{r}\right).$$

$$\text{Let}\quad a=\sqrt{2},\quad b=1;$$

in which case the equation for determining x, y, z becomes

$$x^3-(1+v^2)\,x-3v=0.$$

This is satisfied by the following values:

$$\begin{array}{llll} x=1, & y=2.5414, & z=-3.5414, & r=3v=-9, \\ x'=1.05, & y'=2.3974, & z'=-3.4474, & r'=-8.6778. \end{array}$$

Approximating to the value of the arc subtended by the portion of the abscissæ $x'-x$, and calling it arc (x), we find

$$\text{arc}\,(x)=.0721.$$

And similarly,

$$\text{arc}\,(y)=.1364,$$
$$\text{arc}\,(z)=.0907.$$

Now, for a reason which we will state presently, arc (x) must be considered negative. Therefore we have

$$\begin{aligned} \text{arc}\,(y)+\text{arc}\,(z) &= .2271 \\ -\,\text{arc}\,(x) &= .0721 \\ \text{sum} &= .1550 \end{aligned}$$

Also,

$$\frac{3}{2}\cdot\frac{r'-r}{a^2+b^2}=\frac{1}{2}(r'-r)=\quad .1611$$

$$\frac{a^4-b^4}{2}\left(\frac{1}{r'}-\frac{1}{r}\right)=\frac{3}{2}\left(\frac{1}{r'}-\frac{1}{r}\right)=-\,.0061$$

$$\text{sum}=\quad .1550$$

which agrees with the theorem.

Mr. Talbot's reasoning with regard to the signs of the arcs is of this kind. Taking the example before us, since x and y are both positive and z negative, vx or $\frac{x^2yz}{3}$ is negative; therefore its equivalent

$$\sqrt{x^4 - 2m^2x^2 \cos a + m^4} - m^2 \text{ or } \sqrt{x^4 - x^2 + \frac{9}{4}} - \frac{3}{2}$$

must be negative; hence, since the values of x are between 1 and 1.05, the radical must have a negative sign. Similar reasoning will shew that $\sqrt{y^4 - y^2 + \frac{9}{4}}$ must have a negative, and $\sqrt{z^4 - z^2 + \frac{9}{4}}$ a positive sign. Therefore the three integrals must be written

$$-\int \frac{dx}{x^2}\sqrt{x^4 - x^2 + \tfrac{9}{4}}, \quad -\int \frac{dy}{y^2}\sqrt{y^4 - y^2 + \tfrac{9}{4}}, \quad +\int \frac{dz}{z^2}\sqrt{z^4 - z^2 + \tfrac{9}{4}}.$$

Finally, since y' is less than y,

$$-\int_y^{y'} \frac{dy}{y^2}\sqrt{y^4 - y^2 + \tfrac{9}{4}}$$

is positive. Hence the first arc must be considered negative, and the second and third positive.

Q.

VII.—ON THE ACHROMATISM OF EYE-PIECES OF TELESCOPES AND MICROSCOPES.*

Mr. Airy, in a paper published in the Second Volume of the *Cambridge Transactions*, has investigated the conditions under which a system of lenses is achromatic, *per se*—that is, when only one kind of glass is made use of. The enquiry, on account of its immediate application to the eye-pieces of telescopes and microscopes, is one of considerable importance; and as the way in which it is conducted in the paper just mentioned appears to be more complicated than necessary, and does not lead to the most general solution of the problem, perhaps the following attempt may be not wholly without interest.

The difficulty of the question consists in finding the angle which a ray of light makes with the axis of the system of lenses after having passed through it. When this is done, we have only to take the chromatic variation, and equate it to zero, to get the general equation of achromatism for the system.

In what follows, lines are considered positive when measured in a direction opposite to that of the incident ray.

Let y_n be the tangent of the angle which the ray makes with the

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* From a Correspondent.

axis before its incidence on the n^{th} lens of the system; let z_n be the distance from the axis of the point where it impinges on that lens; and take a_{n-1} to signify the distance of the n^{th} from the $n-1^{\text{th}}$ lens. Then $\frac{z_n}{y_n}$ and $\frac{z_n}{y_{n+1}}$ are the distances of the conjugate foci from the n^{th} lens, and by the ordinary formula

$$\frac{y_{n+1}}{z_n} - \frac{y_n}{z_n} = \rho_n,$$

ρ being the reciprocal of the focal length; therefore

$$y_{n+1} - y_n = \rho_n z_n.$$

Again, we have the simply geometrical relation

$$\frac{z_{n+1}}{y_{n+1}} - \frac{z_n}{y_{n+1}} = a_n,$$

$$\text{or} \quad z_{n+1} - z_n = a_n y_{n+1}.$$

The advantage gained by introducing y_n is, that we thus have precisely similar equations for the optical and geometrical conditions of the problem. Nothing is now easier than by successive substitutions to determine the value of y_n in terms of y and z, and y_n is the tangent of the "visual angle," which we are seeking. As an instance, let it be proposed to determine the conditions of achromatism in a system of three lenses. Mr. Airy has done this only for rays originally parallel to the axis: the method here proposed applies with equal facility to the general case.

The equations required are these

$$y_2 - y_1 = \rho_1 z_1$$

$$z_2 - z_1 = a_1 y_2$$

$$y_3 - y_2 = \rho_2 z_2$$

$$z_3 - z_2 = a_2 y_3$$

$$y_4 - y_3 = \rho_3 z_3$$

Hence

$$y_2 = y_1 + \rho_1 z_1$$

$$z_2 = (1 + a_1\rho_1) z_1 + a_1 y_1$$

$$y_3 = [1 + a_1\rho_2] y_1 + [\rho_1 + \rho_2 + a_1\rho_1\rho_2] z_1$$

$$z_3 = (1 + a_1\rho_1 + a_2\rho_1 + a_2\rho_2 + a_1a_2\rho_1\rho_2) z_1 + (a_1 + a_2 + a_1a_2\rho_2) y_1$$

$$y_4 = [1 + a_1\rho_2 + a_1\rho_3 + a_2\rho_3 + a_1a_2\rho_2\rho_3] y_1 + [\rho_1 + \rho_2 + \rho_3 + a_1\rho_1\rho_2 + a_1\rho_1\rho_3 + a_2\rho_1\rho_3 + a_2\rho_2\rho_3 + a_1a_2\rho_1\rho_2\rho_3] z_1.$$

Taking the chromatic variations of the two terms in the usual way, and equating each to zero, we find

$$a_1\rho_2 + a_1\rho_3 + a_2\rho_3 + 2a_1a_2\rho_2\rho_3 = 0$$

$$\rho_1 + \rho_2 + \rho_3 + 2a_1\rho_1\rho_2 + 2a_1\rho_1\rho_3 + 2a_2\rho_1\rho_3 + 2a_2\rho_2\rho_3 + 3a_1a_2\rho_1\rho_2\rho_3 = 0.$$

The first of these equations becomes unnecessary in the particular case considered by Mr. Airy, viz. that in which $y_1 = 0$; the

second is identical with that given by him at p. 245, when attention is paid to the signs. Taken together they determine the relative positions of three given lenses, which shall form a combination achromatic for rays of any degree of obliquity.

In the particular case in which the focal distances of all the lenses are equal, and the intervals a_1, a_2, &c. are also equal, the general equations degenerate into a system of simultaneous equations in finite differences. They are then

$$y_{n+1} - y_n = \rho z_n, \quad z_{n+1} - z_n = a y_{n+1}.$$

Eliminating z_n, we get

$$y_{n+2} - (\rho a + 2) y_{n+1} + y_n = 0.$$

The general solution of this will be

$$y_n = c\mathrm{A}^n + c_1 \mathrm{A}^{-n},$$

A being a root of the recurring quadratic equation

$$x^2 - (\rho a + 2) x + 1 = 0,$$

c and c are to be found by the conditions

$$y_1 = c\mathrm{A} + c_1 \mathrm{A}^{-1}, \quad y_1 + \rho z_1 = c\mathrm{A}^2 + c_1 \mathrm{A}^{-2}.$$

The general solution of the system of quasi-equations employed in the enquiry must involve some functional operation which degenerates into the radical contained in A.

It would be perhaps worth considering how far we might be able to present this operation in a distinct form, defined and distinguished by a particular symbol; but the subject is not one which can be discussed at present. At any rate, we see that the research of the general expression for y_n is one of considerable difficulty.

The greater part of the investigation given by Mr. Airy in the conclusion of his paper, with respect to the achromatism of microscopes, becomes unnecessary by employing the general expression given above for y_4. His object is to determine the distance of an object-glass of given focal length from a diaphragm whose distance from the field-glass of a given eye-piece of three lenses is given.

Let a_0 be the distance of the diaphragm from the field-glass; therefore we have $z_1 = a_0 y_1$, and putting this value for z_1, we get an expression for y_4 of the form $y_4 = y_1 \mathrm{R}$. The chromatic variation of this is to be zero, and consequently that of its logarithm;

$$\therefore 0 = \frac{\Delta y_1}{y_1} + \frac{\Delta \mathrm{R}}{\mathrm{R}}.$$

Now Mr. Airy has shown that, (adopting the notation of this paper)

$$\frac{\Delta y_1}{y_1} = -x\rho_0 \frac{\delta\mu}{\mu - 1}$$

x being the distance of the object-glass from the diaphragm, and ρ_0 its vergency, or the reciprocal of its focal length. Putting for R its value, we get at once $x\rho_0 =$

$$[a_0[\rho_1+\rho_2+\rho_3]+a_1[\rho_2+\rho_3]+a_2\rho_3+2a_0a_1[\rho_1\rho_2+\rho_1\rho_3]$$
$$+2a_0a_2[\rho_1\rho_3+\rho_2\rho_3]+2a_1a_2\rho_2\rho_3+3a_0a_1a_2\rho_1\rho_2\rho_3]$$

divided by

$$[1+a[\rho_1+\rho_2+\rho_3]+a_1[\rho_2+\rho_3]+a_2\rho_3+a_0a_1[\rho_1\rho_2+\rho_1\rho_3]$$
$$+a_1a_2[\rho_1\rho_3+\rho_2\rho_3]+a_1a_2\rho_2\rho_3+a_0a_1a_2\rho_1\rho_2\rho_3]$$

which is identical with his result.

R. L. E.

VIII.—ON THE TRANSFORMATION OF HOMOGENEOUS FUNCTIONS OF THE SECOND DEGREE.

(1). A homogeneous function, of the second degree, of any number of variables, may always be transformed into another which shall contain only the squares of the new variables. The method of doing this will be found in a paper by M. Lebesgue, in *Liouville's Journal de Mathématiques*, tom. II. p. 337. The case in which the variables are three in number, and the transformation amounts to a change in the axes of coordinates, frequently occurs. It presents itself in the reduction of the equation to surfaces of the second order; and the properties of principal axes of rotation, of the axes of elasticity of a crystal, &c. may be shown to depend upon the same transformation.

Thus the moment of inertia of a solid about an axis whose direction-cosines are l, m, n, is

$$=\int(x^2+y^2+z^2)\,dM-l^2\int x^2\,dM-m^2\int y^2\,dM-n^2\int z^2\,dM$$
$$-2mn\int yz\,dM-2ln\int xz\,dM-2lm\int xy\,dM\ldots\ldots\ldots(1).$$

The form of the body being known, the integrals are given quantities, which we may write A, B, C, f, g, h.

The first term is independent of the direction of the axes. The remainder of the expression, changing the signs, is

$$Al^2+Bm^2+Cn^2+2fmn+2g\,ln+2h\,lm\ldots\ldots(2).$$

If x_1, y_1, z_1 be the coordinates of a point referred to new axes, l_1, m_1, n_1 the direction-cosines of the axis of inertia, then, by the known formulæ,

$$\left.\begin{aligned}x&=ax_1+a'y_1+a''z_1\\ y&=bx_1+b'y_1+b''z_1\\ z&=cx_1+c'y_1+c''z_1\end{aligned}\right\}\ldots(3),\qquad \left.\begin{aligned}l&=al_1+a'm_1+a''n_1\\ m&=bl_1+b'm_1+b''n_1\\ n&=cl_1+c'm_1+c''n_1\end{aligned}\right\}\ldots(4).$$

Between the 9 quantities a, b, c; a', b', c'; a'', b'', c'', we have the 6 relations

$$\left.\begin{aligned} a'a''+b'b''+c'c''&=0\\ a''a+b''b+c''c&=0\\ aa'+bb'+cc'&=0\end{aligned}\right\}\ldots(5),\qquad \left.\begin{aligned} a^2+b^2+c^2&=1\\ a'^2+b'^2+c'^2&=1\\ a''^2+b''^2+c''^2&=1\end{aligned}\right\}\ldots(6).$$

By the substitution of the values of l, m, n from formulæ (4), (2) takes the form

$$Pl_1^2+Qm_1^2+Rn_1^2+2P'm_1n_1+2Q'l_1n_1+2R'l_1m_1\ldots(7),$$

the quantities PQR, P′Q′R′ being functions of

$$abc,\ a'b'c',\ a''b''c'',\ ABC,\ fgh.$$

If we represent $\int x_1^2\,dm$, &c., by A′, &c. this expression is equivalent to

$$A'l_1^2+B'm_1^2+C'n_1^2+2f'm_1n_1+2g'l_1n_1+2h'l_1m_1\ldots(8);$$

we thus find $f'=P'$, that is, $\int y'z'\,dm$ expressed in terms of

$$abc\ a'\ \ldots\ A\ \ldots\ldots$$

and therefore, if the new axes are so chosen that P′Q′R′, the coefficients of the products, vanish in the transformed expression (2), these new axes will possess the property of making

$$\int x_1y_1\,dm=0,\quad \int x_1z_1\,dm=0,\quad \int y_1z_1\,dm=0,$$

and are consequently principal axes of rotation.

(2). In finding the axes of elasticity of a crystal, we require a transformation of coordinates which will make

$$\frac{d^2U}{dx_1dy_1}=0,\quad \frac{d^2U}{dx_1dz_1}=0,\quad \frac{d^2U}{dy_1dz_1}=0,$$

U being a known function of x, y, and z: this amounts to finding the linear substitution for dx, dy, dz, which will cause the terms involving dx_1dy_1, dx_1dz_1, dy_1dz_1 to vanish in the transformed expression for

$$\frac{d^2U}{dx^2}dx^2+\frac{d^2U}{dy^2}dy^2+\frac{d^2U}{dz^2}dz^2+2\frac{d^2U}{dy\,dz}dy\,dz+2\frac{d^2U}{dx\,dz}dx\,dz$$
$$+2\frac{d^2U}{dx\,dy}dx\,dy.$$

(3). The well-known formulæ for the transformation of coordinates, contain either three arbitrary quantities independent of each other, or nine arbitrary quantities, with six relations between them: the use of the first set is embarrassing, from the complexity of the expressions and the want of any evident law in their formation; and even when the transformation is made by two successive steps, the calculations, though less embarrassing, are still, as may be seen from a preceding article in our present Number, both tedious and awkward.

(4). In making use of the symmetrical formulæ, the method which first presents itself is to substitute the expressions for the variables, and perform all the multiplications indicated. But the squares and products of the six trinomials give complex expressions.

The conditions to be fulfilled give equations involving the nine arbitrary quantities, and it is not easy to see how the eliminations are to be effected, and the value of each determined.

(5). This difficulty may be avoided in different ways. In the reduction of the equation of surfaces of the second order, by the consideration of diametral planes which are perpendicular to their chords, the transformation is performed by two successive operations. The consideration, that at the extremities of the principal axes, if such exist, the tangent plane must be perpendicular to the axes, and therefore the radius vector a maximum or minimum, leads to a very simple investigation, which will be found in the second Number of this Journal, p. 53.

The other problems which we have mentioned, are better solved from independent considerations, by which the conditions and the values of the coefficients are more easily obtained: it is, however, of importance to show, as we have done, that they depend essentially on the same transformation, and so account for the occurrence in every method of the final cubic equation. For these methods we may refer to the first Number of the Journal, pp. 4—35, and the latter has been inserted in the second edition of a *Treatise on Dynamics*, by Mr. Earnshaw, of St. John's College.

(6). Taking, then, the general form of a homogeneous function of the second degree in x, y, z,

$$Ax^2+By^2+Cz^2+2fyz+2gxz+2hxy\dots\dots\dots(9).$$

This may be put under the form

$$(Ax+hy+gz)\,x+(hx+By+fz)\,y+(gx+fy+Cz)\,z\ ..(10),$$

and substituting for x, y, z the values (3), this becomes

$$\left.\begin{aligned}&(Lx_1+L'y_1+L''z_1)\,(ax_1+a'y_1+a''z_1)\\+&(Mx_1+M'y_1+M''z_1)\,(bx_1+b'y_1+b''z_1)\\+&(Nx_1+N'y_1+N''z_1)\,(cx_1+c'y_1+c''z_1)\end{aligned}\right\}\dots(11).$$

In which

$$\left.\begin{aligned}L^{(m)}&=Aa^{(m)}+hb^{(m)}+gc^{(m)}\\M^{(m)}&=ha^{(m)}+Bb^{(m)}+fc^{(m)}\\N^{(m)}&=ga^{(m)}+fb^{(m)}+Cc^{(m)}\end{aligned}\right\}\ ..(12),$$

(11) may be thus put under the same form as (10),

$$\begin{aligned}(Px_1+P'y_1+P''z_1)\,x_1+(P_{,}x_1+P_{,}'y_1+P_{,}''z_1)\,y_1&\\+(P_{,,}x_1+P_{,,}'y_1+P_{,,}''z_1)\,z_1&\dots(13).\end{aligned}$$

In which

$$P_{(n)}{}^{(m)}=a^{(n)}L^{(m)}+b^{(n)}M^{(m)}+c^{(n)}N^{(m)}\dots\dots(14);$$

and from the form of expression (12) it will be seen that

$$P_{(n)}{}^{(m)}=P_{(m)}{}^{(n)},$$

$$\text{and therefore } P'=P_{,},\quad P''=P_{,,},\quad P_{,}''=P_{,,}';$$

and therefore the products will vanish if these quantities are each equal to zero.

$$\text{Now,}\qquad \left.\begin{aligned} P &= aL + bM + cN \\ P_{\prime} = 0 &= a'L + b'M + c'N \\ P_{\prime\prime} = 0 &= a''L + b''M + c''N \end{aligned}\right\} \ldots (15).$$

If we multiply these by a, a', a'', and add,

$$aP = L, \text{ and therefore } = Aa + hb + gc,$$

$$\left.\begin{aligned} &\text{or } (A - P)a + hb + gc = 0. \\ &\text{If we multiply by } b, b', b'', \ c, c', c'', \text{ we shall get} \\ &ha + (B - P)b + fc = 0, \\ &ga + fb + (C - P)c = 0, \end{aligned}\right\} \ldots (16),$$

and eliminating a, b and c by cross multiplication,

$$(A-P)(B-P)(C-P) - f^2(A-P) - g^2(B-P) - h^2(C-P) + 2fgh = 0$$
$$= (\text{say})\ U \ldots\ldots (17),$$

(7). This cubic equation, which constantly occurs in mechanics and geometry, has all its roots real. This will be evident if we put it under the form

$$\{(A-P)(B-P) - h^2\}\{(A-P)(C-P) - g^2\} - \{(P-A)f + gh\}^2 = 0,$$

by introducing the factor $(A - P)$.

Either of the two first factors, equated to zero, will give two real values of $A - P$, one positive the other negative, since the last terms are essentially negative; call these $-\rho$ and σ.

If in the left-hand side of the equation,

for $A - P$ we substitute $-\infty$, the result is $+$,
... ... $-\rho$, ... $-$,
... ... 0, ... 0,
... ... σ, ... $-$,
... ... $+\infty$, ... $+$.

Hence there is one real root $< -\rho$, another $> \sigma$; and as there is one $= 0$ between $-\rho$ and σ, and no change of sign, there must also be another; and hence all the roots are real.

(8). We should get the same cubic in finding the values of $a'b'c'$ and $a''b''c''$, as we evidently ought, since there is nothing to determine which of the three directions should be taken for the axis of x, or y, or z.

Let the three roots be P_1, P_2, P_3; it is evident we must take different roots according as we wish to find one of the sets abc, $a'b'c'$, or $a''b''c''$.

(9). To find the actual values of a, b, c, eliminate b between the first and second, and then between the second and third of equations (16): we thus get

$$a\{(A-P)(B-P)-h^2\} = c\{fh - g.(B-P)\},$$

$$a\{fh - g.(B-P)\} = c\{(B-P)(C-P)-f^2\},$$

whence $$\frac{a^2}{(B-P)(C-P)-f^2} = \frac{c^2}{(A-P)(B-P)-h^2}$$

$$= \frac{b^2}{(A-P)(C-P)-g^2} = \text{(say) } \mu.$$

Hence $$a^2+b^2+c^2=\mu\{(A-P)(B-P)+(A-P)(C-P)$$

$$+(B-P)(C-P)-f^2-g^2-h^2\}$$

$$= -\mu\frac{dU}{dP};$$

and if for P we put P_1, one of the roots,

$$1 = \mu.(P_1 - P_2)(P_1 - P_3);$$

therefore $$a^2 = \frac{(B-P)(C-P)-f^2}{(P_1-P_2)(P_1-P_3)},$$

with similar expressions for b'^2, c'^2, a'^2, &c.

(10). The actual values of a, b, c are seldom required; in general it is sufficient to assure ourselves of the possibility of the transformation. The preceding investigation shows that it is always possible, and accounts for the constant occurrence of the equation $U = 0$, whenever the rectangularity of three lines is to be expressed.

H. T.

IX.—INVESTIGATION OF THE TENDENCY OF A BEAM TO BREAK WHEN LOADED WITH WEIGHTS.

In the following elementary Statical Problem, a discontinuous function presents itself, which admits of a very simple geometrical representation.

A beam, whose length is l, is supported horizontally at its extremities A and B. At distances a and b from these extremities is placed a weight W: and it is required to find the *tendency* of the beam to break at any point.

Let A and B represent the pressures at the extremities. Take $AP = x$, and suppose the weight of the beam to be neglected.

The tendency to break will be found by supposing one end of the beam to be fixed, say built into a wall, and finding the force tending to *turn* the other end about the point in consideration; we have thus, calling the tendency to break at P, T_x,

$$T_x = Ax, \text{ or } = B(l-x) - W(a-x),$$
$$= \frac{Wb}{l} \cdot x \text{ .... (1).}$$

It is evident that there is no tendency to break at the extremities; accordingly we find from the expression for T_x, $T_0 = 0$, but for B we have $T_l = Wl$, which it is evident cannot be the case. In fact, the function representing the tendency to break at any point is *discontinuous*, and changes its form at the point W.

If x be greater than a, we find

$$T_x = Ax - W(x-a) = B(l-x),$$
$$= \frac{Wa}{l}(l-x) \text{ .... (2).}$$

When $x=a$, the formulæ (1) and (2) agree in giving $T_a = \frac{Wab}{l}$.

If we take a line WC perpendicular to the beam to represent this, and join CA and CB, the tendency to break at any point P is represented by the ordinate to the broken line ACB.

We may represent the value of T_x, analytically, by help of discontinuous factors, whose values are unity or zero as x is greater or less than a.

$$\frac{1}{1+0^{a-x}} \text{ and } \frac{1}{1+0^{x-a}}$$

are of this nature, and have the values 1, $\frac{1}{2}$, 0, as $x-a$ or $a-x$ are respectively positive, zero, or negative; so that

$$T_x = \frac{W}{l} \cdot \left\{\frac{bx}{1+0^{a-x}} + \frac{a(l-x)}{1+0^{x-a}}\right\},$$
$$= \frac{W}{l} \cdot \frac{1}{0^a+0^x}\{bx\,0^x + a(l-x)\,0^a\}.$$

If we suppose the section and density of the beam y and ρ to vary according to any law; then the tendency to break is

$$T_x = Ax - g\int_0^x \rho y(x-z)\,dz;$$

when $x = l$, $T_l = 0$,

$$0 = Al - g\int_0^l \rho y(l-z)\,dz;$$

$$\therefore T_x = g\frac{x}{l}\int_0^l \rho y(l-z)\,dz - g\int_0^x \rho y(x-z)\,dz,$$
$$= g\frac{l-x}{l}\int_0^x \rho\,yz\,dz + g\frac{x}{l}\int_x^l \rho\,y(l-z)\,dz.$$

If the beam is loaded with n weights $W_1, W_2 \ldots W_n$, placed at distances $x_1 \ldots x_n$ from A, and x lies between x_r and x_{r+1}, then it will be seen that we must add to the value of T_x,

$$\frac{l-x}{l}\sum_1^r W_s x_s + \frac{x}{l}\sum_{r+1}^r W_s(l-x_s).$$

The first part of the expression will represent a curve, which may be constructed on one side of AB; the second will represent the sides of a polygon, constructed also upon AB; and the tendency to break will be given by the sum of the ordinates to the two lines.

We may, as before, represent the complete value of T_x, which will be

$$T_x = g\frac{l-x}{l}\int_0^l \frac{\rho yz\,dz}{1+0^{x-z}} + g\frac{x}{l}\int_0^l \frac{\rho y(l-z)\,dz}{1+0^{z-x}}$$

$$+\frac{l-x}{l}\sum_1^n \frac{W_s x_s}{1+0^{x-x_s}} + \frac{x}{l}\sum_1^n \frac{W_s(l-x_s)}{1+0^{x_s-x}}$$

$$=\frac{g}{l}\int_0^l \frac{\rho y}{0^x+0^z}\cdot\{(l-x)z0^z + x(l-z)0^x\}\,dz$$

$$+\frac{1}{l}\sum_l^n \frac{W_s}{0^x+0^{x_s}}\{(l-x)x_s 0^{x_s} + x(l-x_s)0^x\}.$$

H. T.

X.—MATHEMATICAL NOTES.

1*. THE Solution of the Linear Equation of the n^{th} order contained in the First Number of this Journal, fails in a case which is not there adverted to. The instance alluded to is, when the equation $f\left(\frac{d}{dx}\right)=0$ has equal roots. In this case we need not reject the general method, but in applying it we must adopt the following modification.

Suppose the equation to have r roots equal to a, and all the rest unequal; then $f\left(\frac{d}{dx}\right)y = X$ is equivalent to

$$\left(\frac{d}{dx}-a_1\right)\left(\frac{d}{dx}-a_2\right)-\left(\frac{d}{dx}-a_{n-r}\right)\left(\frac{d}{dx}-a\right)^r y = X;$$

whence we may obtain, by the general method,

* From a Correspondent.

$$\left(\frac{d}{dx} - a\right)^r y = \frac{\varepsilon^{a_1 x}\left(\int \varepsilon^{-a_1 x} X\, dx\right)}{(a_1 - a_2)(a_1 - a_3) \dots (a_1 - a_{n-r})}$$
$$+ \frac{\varepsilon^{a_2 x}\left(\int \varepsilon^{-a_2 x} X\, dx\right)}{(a_2 - a_1)(a_2 - a_3) \dots (a_2 - a_{n-r})}$$
$$+ \dots + \frac{\varepsilon^{a_{n-r} x}\left(\int \varepsilon^{-a_{n-r} x} X\, dx\right)}{(a_{n-r} - a_1)(a_{n-r} - a_2) \dots (a_{n-r} - a_{n-r-1})}.$$

Putting the second side equal to X_1, we have

$$y = \left(\frac{d}{dx} - a\right)^r X_1$$
$$= \varepsilon^{ax} \int^r dx^r\, \varepsilon^{-ax} X_1.$$

The complementary function to be added will be found by making $X = 0$, when we get

$$y = \varepsilon^{ax} \int^r dx^r\, 0$$
$$= \varepsilon^{ax}(C_1 x^{r-1} + C_2 x^{r-2} + \&c. + C_{r-1}).$$

λ.

2. *Addendum to Art. 4. of No. V.*—The following transformation may be added to those given in page 217. Take the series

$$S = AA_1 + BB_1 x + CC_1 x^2 + \&c.$$
and let $B = DA, \quad C = D^2 A, \ \&c.$
$$B_1 = D_1 A_1, \quad C_1 = D_1^2 A_1, \ \&c.$$

then the expression becomes

$$S = (1 + xDD_1 + x^2 D^2 D_1^2 + \&c.)\, AA_1$$
$$= (1 - xDD_1)^{-1} AA_1 = \{1 - xD(1 + \Delta_1)\}^{-1} AA_1$$
$$= \{1 - D(x + x\Delta_1)\}^{-1} AA_1 = \varepsilon^{x\Delta_1 \frac{d}{dx}} (1 - xD)^{-1} AA_1.$$

Now $(1 - xD)^{-1} A = A + Bx + Cx^2 + \&c. = X$ suppose; therefore the series becomes

$$S = \varepsilon^{x\Delta_1 \frac{d}{dx}} A_1 X = A_1 X + \frac{x\Delta A_1}{1} \frac{dX}{dx} + \frac{x^2 \Delta^2 A_1}{1.2} \frac{d^2 X}{dx^2} + \&c.$$

φ.

3. The investigation of the locus of a straight line, which rests constantly on three given straight lines, as given by Leroy in page 98 of his *Analytical Geometry*, may be much simplified by using the symmetrical equations to the straight line. Taking the same notation as Leroy, the equations to the directrices are

$$\left.\begin{array}{l} x = +\alpha \\ y = -\beta \end{array}\right\}\dots(1), \quad \left.\begin{array}{l} z = +\gamma \\ x = -\alpha \end{array}\right\}\dots(2), \quad \left.\begin{array}{l} y = +\beta \\ z = -\gamma \end{array}\right\}\dots(3),$$

and the equations to the generatrix are

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n};$$

between which we have to eliminate a, b, c, l, m, n.

The conditions that the generatrix shall pass through (1), (2), (3) respectively, give us the equations

$$\frac{x-\alpha}{l}=\frac{y+\beta}{m},$$

$$\frac{z-\gamma}{n}=\frac{x+\alpha}{l},$$

$$\frac{y-\beta}{m}=\frac{z+\gamma}{n}.$$

Multiplying these equations together, l, m, n are eliminated, and we have

$$(x-\alpha)(y-\beta)(z-\gamma)=(x+\alpha)(y+\beta)(z+\gamma),$$

which reduces to

$$\alpha yz+\beta xz+\gamma xy+\alpha\beta\gamma=0,$$

the equation to the hyperboloid of one sheet.

ϕ.

4*. I propose to exhibit the solution of the partial differential equation

$$\frac{d^n z}{dx^n}+\mathrm{A}_1\frac{d^n z}{dx^{n-1}dy}+\mathrm{A}_2\frac{d^n z}{dx^{n-2}dy}+\ldots+\mathrm{A}_n\frac{d^n z}{dy^n}=\mathrm{V},$$

where A_1, A_2.. A_n are constant, and V any function of x and y. The method I shall follow will bear a strong analogy to that applied to the linear equation between two variables; but it will be requisite to go into it more fully than might at first sight appear necessary.

Separating the symbols we shall have

$$z=\left(\frac{d}{dx}-a_2\frac{d}{dy}\right)^{-1}\left(\frac{d}{dx}-a_2\frac{d}{dy}\right)^{-1}\ldots\left(\frac{d}{dx}-a_n\frac{d}{dy}\right)^{-1}\mathrm{V}.$$

Where a_1, a_2 ... a_n are the roots of the equation,

$$a^n+\mathrm{A}_1 a^{n-1}+\ldots+\mathrm{A}_n=0\ \ldots\ldots\ (a),$$

$$\therefore z=\left(\frac{d}{dx}-a_2\frac{d}{dy}\right)^{-1}\left(\frac{d}{dx}-a_3\frac{d}{dy}\right)^{-1}\ldots\left(\frac{d}{dx}-a_n\frac{d}{dy}\right)^{-1}\varepsilon^{a_1 x\frac{d}{dy}}\int\varepsilon^{-a_1 x}\mathrm{V}dx$$

$$=\left(\frac{d}{dx}-a_3\frac{d}{dy}\right)^{-1}\ldots\left(\frac{d}{dx}-a_n\right)^{-1}\left\{\frac{\varepsilon^{a_1 x\frac{d}{dy}}\int\varepsilon^{-a_1 x\frac{d}{dy}}\mathrm{V}dx}{(a_1-a_2)\frac{d}{dy}}\right.$$

$$\left.+\frac{\varepsilon^{a_2 x\frac{d}{dy}}\int\varepsilon^{-a_2 x\frac{d}{dy}}\mathrm{V}dx}{(a_1-a_2)\frac{d}{dy}}\right\}\ldots\ldots\ldots(1);$$

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* From a Correspondent.

$$\therefore \frac{dz}{dy} = \left(\frac{d}{dx} - a_3 \frac{d}{dy}\right)^{-1} \cdot\cdot \left(\frac{d}{dx} - a_n \frac{d}{dy}\right)^{-1}$$

$$\left(\frac{\varepsilon^{a_1 x}\int \varepsilon^{-a_1 x \frac{d}{dy}} \mathrm{V} dx}{a_1 - a_2} + \frac{\varepsilon^{a_2 x \frac{d}{dy}} \int \varepsilon^{-a_2 x \frac{d}{dy}} \mathrm{V} dx}{a_1 - a_2}\right) \ldots\ldots(2);$$

and proceeding in this manner we shall finally arrive at the equation

$$\frac{d^{n-1}z}{dy^{n-1}} = \frac{\varepsilon^{a_1 x \frac{d}{dy}} \int \varepsilon^{-a_1 x \frac{d}{dy}} \mathrm{V} dx}{(a_1 - a_2)(a_1 - a_3) \ldots (a_1 - a_n)}$$

$$+ \frac{\varepsilon^{a_2 x \frac{d}{dy}} \int \varepsilon^{-a_2 x \frac{d}{dy}} \mathrm{V} dx}{(a_2 - a_1)(a_2 - a_3)..(a_2 - a_n)} + .. + \frac{\varepsilon^{a_n x \frac{d}{dy}} \int \varepsilon^{-a_n x \frac{d}{dy}} \mathrm{V} dx}{(a_n - a_1)(a_n - a_2)..(a_n - a_{n-1})} \ldots (n).$$

$$\text{Let } \int \varepsilon^{-ax \frac{d}{dy}} \mathrm{V} dx = \varepsilon^{-ax} \mathrm{V}_1 + f(y),$$

where f denotes an arbitrary function; and let the denominators of the above fractions be represented by $\frac{1}{\mathrm{C}^1}, \frac{1}{\mathrm{C}^2} \cdots \frac{1}{\mathrm{C}^n}$, then

$$\frac{d^{n-1}y}{dy^{n-1}} = \mathrm{C}_1\mathrm{V}_1 + \mathrm{C}_2\mathrm{V}_2 + \ldots + \mathrm{C}_n\mathrm{V}_n + \mathrm{C}_1\, \varepsilon^{a_1 x \frac{d}{dy}} f_1(y)$$

$$+ \mathrm{C}_2\, \varepsilon^{a_2 x \frac{d}{dy}} f_2(y) + \ldots + \mathrm{C}_n\, \varepsilon^{a_n x \frac{d}{dy}} f_n(y);$$

and observing that $\varepsilon^{a_1 x \frac{d}{dy}} f(y)$ is equivalent to $f(y + ax)$, we shall have, integrating,

$$z = \mathrm{C}_1 \int \mathrm{V}_1 dy^{n-1} + \mathrm{C}_2 \int \mathrm{V}_2 dy^{n-1} + \ldots + \mathrm{C}_n \int \mathrm{V}_n dy^{n-1}$$
$$= \phi_1 (y + a_1 x) + \phi_2 (y + a_2 x) + \ldots + \phi_n (y + a_n x),$$

where ϕ denotes an arbitrary function.

It is most important to observe, that in integrating the expression for $\frac{d^{n-1}z}{dy^{n-1}}$ we are not at liberty to introduce any arbitrary functions of x, as will appear from the following considerations. In passing from equation (1) to equation (2), no arbitrary function of x is lost; and a similar remark applies to each succeeding pair of the equations (1), (2) ... (n): so that in passing from the expression for z to that for $\frac{d^{n-1}z}{dy^{n-1}}$ no arbitrary function of x is lost, and consequently, in passing back again from the expression for $\frac{d^{n-1}z}{dy^{n-1}}$ no arbitrary function of x must be introduced.

If the roots of the equation (a) are not all unequal, there must be a modification of the above process, similar to that suggested above in the case of the linear equation between two variables.

5. To find the product of the differential factors

$$\frac{d}{dx}\left(\frac{d}{dx}-1\right)\left(\frac{d}{dx}-2\right)\ldots\ldots\left\{\frac{d}{dx}-(n-1)\right\}.$$

By the theorem in page 25, we have generally

$$\left(\frac{d}{dx}-a\right)^n = \varepsilon^{ax}\left(\frac{d}{dx}\right)^n \varepsilon^{-ax};$$

therefore

$$\left(\frac{d}{dx}-1\right)\frac{d}{dx} = \varepsilon^{x}\frac{d}{dx}\varepsilon^{-x}\frac{d}{dx},$$

$$\left(\frac{d}{dx}-2\right)\left(\frac{d}{dx}-1\right)\frac{d}{dx} = \varepsilon^{2x}\frac{d}{dx}\varepsilon^{-x}\frac{d}{dx}\varepsilon^{-x}\frac{d}{dx},$$

$$\left(\frac{d}{dx}-3\right)\left(\frac{d}{dx}-2\right)\left(\frac{dx}{dx}-1\right)\frac{d}{dx} = \varepsilon^{3x}\frac{d}{dx}\varepsilon^{-x}\frac{d}{dx}\varepsilon^{-x}\frac{d}{dx}\varepsilon^{-x}\frac{d}{dx},$$

and so on, so that

$$\left\{\frac{d}{dx}-(n-1)\right\}\left\{\frac{d}{dx}-(n-2)\right\}\ldots\left(\frac{d}{dx}-1\right)\frac{d}{dx}$$

$$= \varepsilon^{(n-1)x}\frac{d}{dx}\varepsilon^{-x}\,dx \ldots (n \text{ factors}),$$

$$= \varepsilon^{nx}\left(\varepsilon^{-x}\frac{d}{dx}\right)^n,$$

$$= y^n\left(\frac{d}{dy}\right)^n \text{ if } x = \log y.$$

This theorem may be otherwise expressed—that

$$y^n\left(\frac{d}{dy}\right)^n = y\frac{d}{dy}\left(y\frac{d}{dy}-1\right)\left(y\frac{d}{dy}-2\right)\ldots\left\{y\frac{d}{dy}-(n-1)\right\}.$$

This affords an easy demonstration of a problem in the Senate-House papers for 1839.

Since $x^n = (x+0)^n = (1+\Delta)^x 0^n$

$$= x\Delta 0^n + \frac{x(x-1)}{1.2}\Delta^2 0^n + \frac{x(x-1)(x-2)}{1.2.3}\Delta^3 0^n + \ldots$$

for x put $y\dfrac{d}{dy}$; then

$$\left(y\frac{d}{dy}\right)^n = y\frac{d}{dy}\Delta 0^n + y^2\left(\frac{d}{dy}\right)^2\frac{\Delta^2 0^3}{1.2} + y^3\left(\frac{d}{dy}\right)^3\frac{\Delta^3 0^2}{1.2.3} + \ldots$$

which is a general formula for changing the independent variable from x to ε^x.

ϕ.

END OF VOL. I.